Pergamon

Int. J. Solids Structures Vol. 34, Nos 33–34, pp. 4339–4355, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(97)00021–8

TWO-FIELD VARIATIONAL FORMULATIONS FOR THE PROBLEM OF THE BEAM ON A CONTINUOUS ELASTIC SUPPORT

AGOSTINO ANTONIO CANNAROZZI and ALBERTO CUSTODI
DISTART, Università di Bologna, Bologna, Italy

(*Received* 3 *August* 1996; *in revised form* 12 *December* 1996)

Abstract—A mixed and a modified complementary energy variational formulation for the problem of a continuously supported, axially and transversely loaded beam is presented. The beam and the support are assumed to be linear elastic, and the response of the latter is represented by the current two-parameter model. An equilibrium finite element approach is developed on the ground of the second formulation and the results obtained for some test problems are exposed. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The analysis of continuously supported beams is an argument that has been treated in the literature for quite a while (Hetényi, 1946; Vlasov and Leont'ev, 1966). This argument principally concerns the structural, civil and mechanical engineering fields, and many problems can be handled effectively through such an idealization (Selvadurai, 1979; Scott, 1981).

The relationship between beam deformation and the support's response, i.e., the support-beam interaction scheme, is the prerequisite for developing a procedure of analysis of the beam where the support is described in a synthetic way, excluding any direct analysis of its behaviour. In the case of a transversely loaded beam, a general framework for this scheme was given by Kerr. The author assumes the hypothesis of bilateral, linear elastic behaviour of the support, and admits the contact pressure (Kerr, 1964)—or a linear combination of this pressure and its even derivatives (Kerr, 1984)—as a function of a linear combination of the interfacial displacement and its even derivatives, the variable of differentiation being the coordinate along the support's surface. This description includes Winkler's (1867) one-parameter (local) model, and the Filonenko-Borodich (1940), Pasternak (1954) and Vlasov–Leont'ev (1966) two-parameter (nonlocal) foundation models, which improve on Winkler's model by introducing, in the simplest manner, a coupling between adjacent points of the support's surface. In the case of axially loaded beams, the one-parameter model for the support is currently employed (Liu and Meyerhof, 1987; Chin, 1970; Sayegh and Tso, 1988). On the other hand, for the analysis of piles a description of the soil including coupling is achieved by exploiting fundamental solutions of the semiinfinite elastic medium problem (Poulos and Davis, 1980). Indeed, this fact may lead one to believe that it would also be profitable a two-parameter model for the case of an axially loaded beam.

As the problem of the elastic equilibrium of the beam is governed by ordinary differential equations, the strong formulation has, as a rule, been exploited in the procedures of solution exposed in literature (Hetényi, 1946; Vlasov and Leont'ev, 1966; Selvadurai, 1979; Scott, 1981; Soldatos and Selvadurai, 1985; Miranda and Nair, 1966; Randolph and Wroth, 1978; Vaziri and Xie, 1990). Following this trend, analytical, or "exact", displacement functions, which are of an exponential kind, have been applied extensively for developing stiffness matrices and nodal load vectors, employed in the framework of structural matrix analysis (Zhahoua and Cook, 1983; Ting and Mockry, 1984; Eisenberger and Yankelewsky, 1985; Razaqpur, 1986; Eisenberger and Clastornik, 1987; Chiwanga and Valsangkar, 1988; Karamanlidis and Prakash, 1989; Sirosh and Ghali, 1989; Razaqpur

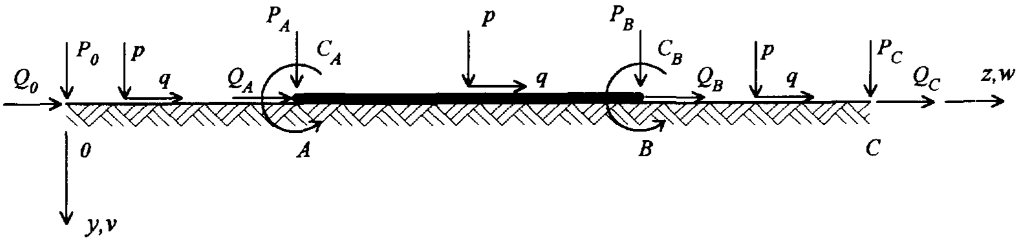


Fig. 1. Scheme of beam and support.

and Shah, 1991; Shirima and Giger, 1992). Obviously, such a modelling makes a mesh refinement unnecessary in order to improve approximation. However, this gain is actual only in the cases ranged by the solutions employed, otherwise it becomes unreal, and the convenience in employing simple algebraic models of representation in conjunction with a direct variational method can have effective results. Indeed, compatible models have been applied in this latter kind of approach (Ting and Mockry, 1984; Eisenberger and Clastornik, 1987; Mourelatos and Parsons, 1987; Yokoyama, 1988; Sayegh and Tso, 1988; Kerr, 1976).

The variational formulations of the problem presented here are in logical correspondence with the displacement variational formulation based on the minimum potential energy principle and provide a basis for developing alternative finite element models. The beam is assumed to be linear elastic, and supposed loaded and deformed in a plane. Axial, bending and shear deformations are taken into account. The support is assumed bilateral, linear elastic, and to react on the beam axially and transversely according to the two-parameter model. The potential function of the support's response is expressed in terms of displacement components and their first derivatives at the interface (Section 2). A mixed functional is derived (Section 3), whose arguments are generalized stresses and displacements for the beam. The variational statement from this functional yields the elasto–kinematic and equilibrium equations in the beam, and compatibility and equilibrium equations at its extremities. Assuming that generalized stresses satisfy, *a priori*, equilibrium in the beam, then a modified complementary energy functional results, whose stationary conditions include the elasto–kinematic equations in an integral form (Section 4). For the sake of completeness, a coherent variational formulation of the response of the support to applied loads is added (Section 5). The latter functional is assumed as the variational basis for a stress equilibrium finite element approach, and the results obtained in some test problems are exposed (Section 6).

2. THE GOVERNING EQUATIONS

Reference is made to a straight beam continuously attached to a support, Fig. 1. The support occupies the interval s, $0 < z < z_C$, of the z-axis, and the beam is placed in the subinterval b, $z_A < z < z_B$, $0 < z_A$, $z_B < z_C$. The ends of the support are free. Beam and support are assumed linear elastic.

The generic cross section of the beam has area $A(z)$ and one of its principal axes is normal to the y–z plane. EA is the axial rigidity of the beam, EI, GA_t are the flexural and shear rigidities in this plane, respectively. $I(z)$ and A_t are cross-sectional moment of inertia and effective shear area, E and G are Young's and shear moduli of the material, respectively.

Support and beam are subjected to the distributed loads $p(z)$ and $q(z)$, in the y and z directions, respectively. The loads P_A, $P_B(P_0, P_C)$ in the y direction, the loads Q_A, $Q_B(Q_0, Q_C)$ in the z direction, and the couples C_A, C_B are lumped at the ends A, B of the beam (O, C of the support), respectively. The displacements in the y and z directions are denoted by $v(z)$ and $w(z)$, $0 \leqslant z \leqslant z_C$. The displacements of support's surface and of the geometrical axis of the beam are admitted to coincide. The flexural rotation in a generic cross section of the beam is denoted by $\varphi(z)$, $z_A \leqslant z \leqslant z_B$. Axial, flexural and shear strains are denoted by ε, χ, γ, in this order, and the compatibility equations† for the beam are:

† A prime denotes differentiation with respect to variable z. The subscript $/u$ means differentiation, or variation, with respect to u.

$$\varepsilon = w', \tag{1_1}$$

$$\gamma = v' + \varphi, \tag{1_2}$$

$$\chi = \varphi'. \tag{1_3}$$

Axial force, shear force and bending moment are denoted by $N(z)$, $V(z)$, $M(z)$, respectively. They are related to the generalized strains by the constitutive equations

$$N = EA\varepsilon \tag{2_1}$$

$$V = GA_t\gamma \tag{2_2}$$

$$M = EI\chi \tag{2_3}$$

and the contact transformations hold

$$N\varepsilon = N^2/(2EA) + EA\varepsilon^2/2 \tag{3_1}$$

$$V\gamma = V^2/(2GA_t) + GA_t\gamma^2/2 \tag{3_2}$$

$$M\chi = M^2/(2EI) + EI\chi^2/2. \tag{3_3}$$

The strain energy density of the support is assumed in the form

$$\psi = \psi_v(v, v') + \psi_w(w, w') \tag{4_1}$$

$$\psi_v = 1/2k_v v^2 + 1/2k_{v1} v'^2 \tag{4_2}$$

$$\psi_w = 1/2k_w w^2 + 1/2k_{w1} w'^2 \tag{4_3}$$

where the elastic moduli k_v, k_{v1}, k_w, k_{w1} are positive, and the following relationships hold

$$2\psi_u = u\psi_{u/u} + u'\psi_{u/u'}, \tag{4_4}$$

$$\psi_{u/u} = u\psi_{u/uu}, \tag{4_5}$$

$$\psi_{u/u'} = u'\psi_{u/u'u'}, \tag{4_6}$$

$$u = v, w.$$

The functional

$$\begin{aligned} TPE(v, w, \varphi) = {} & 1/2 \int_b [EAw'^2 + GA_t(v' + \varphi)^2 + EI\varphi'^2]\,\mathrm{d}z + \\ & \int_s [\psi_v(v, v') + \psi_w(w, w')]\,\mathrm{d}z + \\ & - \int_s (pv + qw)\,\mathrm{d}z - (P_0 v + Q_0 w)_{z=0} - (P_C v + Q_C w)_{z=z_C} + \\ & - (P_A v + Q_A w + C_A\varphi)_{z=z_A} - (P_B v + Q_B w + C_B\varphi)_{z=z_B} \end{aligned} \tag{5}$$

expresses the total potential energy of the system. Its first variation—Appendix A—via the fundamental lemma yields the equations of the elastic equilibrium for the system

• in $s \backslash b$

$$\psi_{v/v} - \psi'_{v/v'} = p \qquad (6_1)$$

$$\psi_{w/w} - \psi'_{w/w'} = q, \qquad (6_2)$$

• at $z = 0$ and $z = z_C$

$$-\psi_{v/v'} = P_0 \qquad (7_1)$$

$$-\psi_{w/w'} = Q_0, \qquad (7_2)$$

$$\psi_{v/v'} = P_C \qquad (8_1)$$

$$\psi_{w/w'} = Q_C, \qquad (8_2)$$

• in b

$$(GA_t(v' + \varphi))' - \psi_{v/v} + \psi'_{v/v'} + p = 0 \qquad (9_1)$$

$$(EAw')' - \psi_{w/w} + \psi'_{w/w'} + q = 0 \qquad (9_2)$$

$$GA_t(v' + \varphi) - (EI\varphi')' = 0, \qquad (9_3)$$

• at $z = z_A$

$$-GA_t(v' + \varphi) + \psi^-_{v/v'} - \psi^+_{v/v'} = P_A \qquad (10_1)$$

$$-EAw' + \psi^-_{w/w'} - \psi^+_{w/w'} = Q_A \qquad (10_2)$$

$$-EI\varphi' = C_A, \qquad (10_3)$$

• at $z = z_B$

$$GA_t(v' + \varphi) + \psi^-_{v/v'} - \psi^+_{v/v'} = P_B \qquad (11_1)$$

$$EAw' + \psi^-_{w/w'} - \psi^+_{w/w'} = Q_B \qquad (11_2)$$

$$EI\varphi' = C_B, \qquad (11_3)$$

where the superscript $^-$ ($^+$) stands for evaluation at $A-0(A+0)$, or at $B-0(B+0)$. Equations (6)–(8) match the support's response and given loads. The elastic equilibrium of the beam (Kerr, 1976) is ruled in b by eqns (9), and at A and B by eqns (10), (11), where the terms with superscript $^-$ in (10), and the ones with superscript $^+$ in (11), express the interaction with the adjacent parts of the support. Functional (5) is the variational basis for the methods of analysis via compatible models.

3. A MIXED VARIATIONAL FORMULATION

The support is assumed to extend under the beam only. Functional (5) becomes

$$\overline{TPE}(v, w, \varphi) = 1/2 \int_b [EAw'^2 + GA_t(v' + \varphi)^2 + EI\varphi'^2]\, dz + \int_b [\psi_v(v, v') + \psi_w(w, w')]\, dz +$$
$$- \int_b (pv + qw)\, dz - (P_A v + Q_A w + C_A \varphi)_{z=z_A} - (P_B v + Q_B w + C_B \varphi)_{z=z_B}. \qquad (12)$$

The generalized displacements at the ends A and B of the beam are denoted by v_A, w_A, φ_A and v_B, w_B, φ_B, respectively. The compatibility conditions at A read

$$v = v_A \tag{13$_1$}$$

$$w = w_A \tag{13$_2$}$$

$$\varphi = \varphi_A, \tag{13$_3$}$$

and similar conditions also hold at extremity B.

A modified Hellinger–Reissner functional is derived from functional (12). The potential energy density of the beam is switched over to the generalized stresses by the application of eqns (1) and (3). Moreover, compatibility conditions (13) at the extremity A, and the analogous ones at the extremity B, are incorporated into the functional by means of Lagrange multipliers $\tilde{V}_A$, $\tilde{N}_A$, $\tilde{M}_A$ and $\tilde{V}_B$, $\tilde{N}_B$, $\tilde{M}_B$, respectively. In this way the following functional is obtained:

$$\begin{aligned} &MHR(V, N, M, v, w, \varphi, \tilde{V}_A, \tilde{N}_A, \tilde{M}_A, \tilde{V}_B, \tilde{N}_B, \tilde{M}_B, v_A, w_A, \varphi_A, v_B, w_B, \varphi_B) = \\ &\quad -1/2 \int_b [N^2/EA + V^2/GA_t + M^2/EI] \, dz + \int_b [Nw' + V(v'+\varphi) + M\varphi'] \, dz + \\ &\quad + \int_b [\psi_v(v, v') + \psi_w(w, w')] \, dz - \int_b (pv + qw) \, dz + \\ &\quad - [(v - v_A)\tilde{V}_A + (w - w_A)\tilde{N}_A + (\varphi - \varphi_A)\tilde{M}_A]_{z=z_A} + \\ &\quad + [(v - v_B)\tilde{V}_B + (w - w_B)\tilde{N}_B + (\varphi - \varphi_B)\tilde{M}_B]_{z=z_B} + \\ &\quad - (P_A v_A + Q_A w_A + C_A \varphi_A) - (P_B v_B + Q_B w_B + C_B \varphi_B). \end{aligned} \tag{14}$$

The first variation of functional MHR—see Appendix A—results in the stationary conditions:

- in b, $z_A < z < z_B$,
  elastokinematic relationships

$$w' - N/EA = 0 \tag{15$_1$}$$

$$(v' + \varphi) - V/GA_t = 0 \tag{15$_2$}$$

$$\varphi' - M/EI = 0, \tag{15$_3$}$$

  equilibrium equations

$$N' - \psi_{w/w} + \psi'_{w/w'} + q = 0 \tag{16$_1$}$$

$$V' - \psi_{v/v} + \psi'_{v/v'} + p = 0 \tag{16$_2$}$$

$$V - M' = 0, \tag{16$_3$}$$

- at extremity A, $z = z_A$,
  compatibility

$$v = v_A \tag{17$_1$}$$

$$w = w_A \tag{17$_2$}$$

$$\varphi = \varphi_A, \tag{17$_3$}$$

  identification of the Lagrange multipliers with the inner force components

$$\tilde{V}_A = -V - \psi_{v/v'} \tag{18$_1$}$$

$$\tilde{N}_A = -N - \psi_{w/w'} \tag{18$_2$}$$

$$\tilde{M}_A = -M, \tag{18$_3$}$$

and with the external loads

$$\tilde{V}_A = P_A \tag{19$_1$}$$

$$\tilde{N}_A = Q_A \tag{19$_2$}$$

$$\tilde{M}_A = C_A, \tag{19$_3$}$$

and analogous expressions at extremity B, $z = z_B$.

After the introduction of identification (18) into functional (12), the resulting functional can be employed in a variational statement for mixed models (Atluri *et al.*, 1983), where the displacements in b are the arguments of the strain energy of the support and play the role of Lagrange multipliers for the weak fulfilment of equilibrium.

4. A MODIFIED COMPLEMENTARY ENERGY FORMULATION

A further, two-field variational formulation can be derived from functional (14) by prescribing that the generalized stresses fulfil eqns (16) in advance.

Axial force N, shear force V and bending moment M are split into three parts

$$N = N_0 + N_q + N_r \tag{20$_1$}$$

$$V = V_0 + V_p + V_r \tag{20$_2$}$$

$$M = M_c + M_p + M_r, \tag{20$_3$}$$

defined as follows:

N_0, V_0, M_c satisfy equilibrium eqns (16), with external loads and support response both equal to zero:

$$N_0' = 0 \tag{21$_1$}$$

$$V_0' = 0 \tag{21$_2$}$$

$$M_c = V_0(z - z_A) + M_0 \tag{21$_3$}$$

i.e., N_0, V_0, M_0 are constant with respect to z, N_q, V_p, M_p are in equilibrium with, and depend only on, the given loads:

$$N_q = -\int_{z_A}^{z} q(\zeta)\,\mathrm{d}\zeta \tag{22$_1$}$$

$$V_p = -\int_{z_A}^{z} p(\zeta)\,\mathrm{d}\zeta \tag{22$_2$}$$

$$M_p = -\int_{z_A}^{z} \mathrm{d}\varsigma \int_0^{\varsigma} p(\zeta)\,\mathrm{d}\zeta, \tag{22$_3$}$$

N_r, V_r are formally defined as the primitives of the support's response, and M_r as the primitive of V_r, as follows:

$$N_r = \int_{z_A}^{z} \psi_{w/w}\,\mathrm{d}\zeta - \psi_{w/w'}(z) + \psi_{w/w'}(z)\,|_{z=z_A} \tag{23$_1$}$$

$$V_r = \int_{z_A}^{z} \psi_{v/v}\,\mathrm{d}\zeta - \psi_{v/v'}(z) + \psi_{v/v'}(z)\,|_{z=z_A} \tag{23$_2$}$$

$$M_r = \int_{z_A}^{z} \mathrm{d}\zeta \int_{z_A}^{\zeta} \psi_{v/v}(\varsigma)\,\mathrm{d}\varsigma - \int_{z_A}^{z} \psi_{v/v'}(\zeta)\,\mathrm{d}\zeta + \psi_{v/v'}(z)\,|_{z=z_A}(z-z_A). \tag{23$_3$}$$

Functional (14) is rewritten in the equivalent form

$$\begin{aligned} MHR = {} & -1/2 \int_b [N^2/EA + V^2/GA_t + M^2/EI]\,\mathrm{d}z + \\ & - \int_b [(N'+q)w + (V'+p)v + (M'-V)\varphi]\,\mathrm{d}z + \int_b [\psi_v(v, v') + \psi_w(w, w')]\,\mathrm{d}z + \\ & - [(\tilde{N}_A + N)w + (\tilde{V}_A + V)v + (\tilde{M}_A + M)\varphi]_{z=z_A} + \\ & + [(\tilde{N}_B + N)w + (\tilde{V}_B + V)v + (\tilde{M}_B + M)\varphi]_{z=z_B} + \\ & - [(Q_A - \tilde{N}_A)w_A + (P_A - \tilde{V}_A)v_A + (C_A - \tilde{M}_A)\varphi_A] + \\ & - [(Q_B + \tilde{N}_B)w_B + (P_B + \tilde{V}_B)v_B + (C_B + \tilde{M}_B)\varphi_B], \end{aligned} \tag{24}$$

and definitions (21)–(23) and identifications (18) are introduced. Integration by parts on the terms of the support and some algebra lead to the functional

$$\begin{aligned} & MCE(V_0, N_0, M_0, v, w, v_A, w_A, \varphi_A, v_B, w_B, \varphi_B) = \\ & \quad -1/2 \int_b [N^2/EA + V^2/GA_t + M^2/EI]\,\mathrm{d}z - \int_b [\psi_v(v, v') + \psi_w(w, w')]\,\mathrm{d}z + \\ & \quad - [(N + \psi_{w/w'} + Q_A)w_A + (V + \psi_{v/v'} + P_A)v_A + (M + C_A)\varphi_A]_{z=z_A} + \\ & \quad + [(N + \psi_{w/w'} - Q_B)w_B + (V + \psi_{v/v'} - P_B)v_B + (M - C_B)\varphi_B]_{z=z_B} \end{aligned} \tag{25}$$

where the displacements in b are simply the arguments of the strain energy of the support. By setting

$$\mathcal{N}(z) = \int_{z_A}^{z} N/EA\,\mathrm{d}z \tag{26$_1$}$$

$$\mathcal{V}(z) = \int_{z_A}^{z} V/GA_t\,\mathrm{d}z \tag{26$_2$}$$

$$\mathcal{M}(z) = \int_{z_A}^{z} M/EI\,\mathrm{d}z, \tag{26$_3$}$$

$$\mathscr{M}(z) = \int_{z_A}^{z} \left(\int_{z_A}^{\zeta} M/EI\,\mathrm{d}\varsigma \right) \mathrm{d}\zeta \tag{26$_4$}$$

the stationary conditions of functional MCE flow from its first variation—Appendix B, the moduli k_v, k_{v1}, k_w, k_{w1} are positive—and are

• with respect to N_0, V_0 and M_0

$$w_B - w_A - \mathcal{N}(z_B) = 0 \tag{27$_1$}$$

$$v_B - v_A + \varphi_B(z_B - z_A) - \mathcal{V}(z_B) + \mathbb{M}(z_B) - \mathcal{M}(z_B)(z_B - z_A) = 0 \tag{27$_2$}$$

$$\varphi_B - \varphi_A - \mathcal{M}(z_B) = 0, \tag{27$_3$}$$

• with respect to w

$$[w - w_B - \mathcal{N} + \mathcal{N}(z_B)]k_w - [(w' - \mathcal{N}')k_{w1}]' = 0 \tag{28$_1$}$$

$$(w' - \mathcal{N}')_{z=z_A} = 0 \tag{28$_2$}$$

$$(w' - \mathcal{N}')_{z=z_B} = 0 \tag{28$_3$}$$

$$w_B - w_A - \mathcal{N}(z_B) = 0, \tag{28$_4$}$$

• with respect to v

$$[\mathbb{M} - \mathbb{M}(z_B) - \mathcal{V} + \mathcal{V}(z_B) - v + v_B - (\varphi_B - \mathcal{M}(z_B))(z_B - z)]k_v + [(\mathcal{V}' - \mathcal{M} + v' - \varphi_B + \mathcal{M}(z_B))k_{v1}]' = 0 \tag{29$_1$}$$

$$[\mathcal{V}' - v' - \varphi_B + \mathcal{M}(z_B)]_{z=z_A} = 0 \tag{29$_2$}$$

$$[\mathcal{V}' - v' - \varphi_B]_{z=z_B} = 0 \tag{29$_3$}$$

$$v_B - v_A + \varphi_B(z_B - z_A) - \mathcal{V}(z_B) + \mathbb{M}(z_B) - \mathcal{M}(z_B)(z_B - z_A) = 0 \tag{29$_4$}$$

• with respect to w_A, v_A and φ_A

$$N + \psi_{w/w'} + Q_A = 0 \tag{30$_1$}$$

$$V + \psi_{v/v'} + P_A = 0 \tag{30$_2$}$$

$$M + C_A = 0, \tag{30$_3$}$$

• with respect to w_B, v_B and φ_B

$$N + \psi_{w/w'} - Q_B = 0 \tag{31$_1$}$$

$$V + \psi_{v/v'} - P_B = 0 \tag{31$_2$}$$

$$M - C_B = 0. \tag{31$_3$}$$

Functional (25) is a modified complementary energy functional for the problem at hand. Equations (27) are the compatibility statement between the whole deformation of the beam and the relative displacements of its extremities—see eqn (B1). Equations (28), (29) stem, in principle, from the stationary conditions with respect to the inner forces which are in equilibrium with the support's response in terms of displacements—see eqns (B2), (B3). This provided, the meaning of compatibility conditions is entirely recognizable for them, even if they apparently descend from the variation with respect to kinematic descriptors. In other terms, eqns (28), (29) mean that the stationary of functional (25) is characterized by the existence of displacements of the support which match with the deformations of the beam, in terms of inner forces, and with the generalized displacements at its extremities. Hence $w(z)$ and $v(z)$ coincide with the displacements in the beam.

Conditions (30) and (31) express boundary equilibrium of the system, beam plus support, in this case—support interrupted at the ends of the beam. It should be noted that eqns ($28_{1,3}$) are a linear, homogeneous, second order differential system in the function in the first square brackets of (28_1). The solution of this problem is

$$w - w_B - \mathcal{N} + \mathcal{N}(z_B) = 0, \tag{32}$$

hence, this condition is equivalent to eqns ($28_{1,3}$).

For the same reason, the condition

$$\mathcal{M} - \mathcal{M}(z_B) - \mathcal{V} + \mathcal{V}(z_B) + v - v_B - (\varphi_B - \mathcal{M}(z_B))(z_B - z) = 0 \tag{33}$$

is equivalent to eqns ($29_{1,3}$).

Conditions (27_1) and (27_2) are repeated in eqns (28_4) (29_4). However, they follow from the variations of functional (25) with respect to the independent arguments, N_0, V_0 and w, v, respectively—see eqns (B1)–(B3) in Appendix B. This apparent prolixity is due to the nonlocal response of the support. Indeed, if parameter k_{w1} were zero—local response—then compatibility would still remain accounted for by means of eqns (27_1), (28_1) and (32), (33), the latter equations being directly obtained from variation (B2). In particular, this fact is a consequence of the simultaneous presence of independent forces N_0, V_0 and $\psi_{w/w'}$, $\psi_{v/v'}$, at the ends of the beam—see eqns ($18_{1,2}$).

All this provided, the effective stationary conditions for functional (25) are eqns (27), (32), (33)—compatibility—and (30), (31)—boundary equilibrium. One may conjecture that the possible application of models of higher degree for the response of the support (Kerr, 1984) would result in a more pronounced formal prolixity.

5. A RELAXED VARIATIONAL FORMULATION FOR THE SUPPORT

A variational formulation, with relaxed continuity requirements, of the support's response to given loads is the logical complement to the previous variational formulation. On this regard, expression (24) is modified by dropping all the terms related to the beam and by replacing the Lagrange multipliers $\tilde{V}_A$, $\tilde{V}_B$ and $\tilde{N}_A$, $\tilde{N}_B$ with $(-\psi_{v/v'})$ and $(-\psi_{w/w'})$, respectively—see identifications (18_1), (18_2).

Referring for simplicity to the interval $0 \leqslant z \leqslant z_C$, Fig. 1, and denoting with v_0, w_0, v_C, w_C the displacements at 0 and C, the following functional results

$$\begin{aligned} MPE_S(v, w, v_0, w_0, v_C, w_C) = {} & \int_0^{z_C} (\psi_v + \psi_w)\,\mathrm{d}z - \int_0^{z_C} (pv + qw)\,\mathrm{d}z + \\ & + [(v - v_0)\psi_{v/v'} + (w - w_0)\psi_{w/w'}]_{z=0} - [(v - v_C)\psi_{v/v'} + (w - w_C)\psi_{w/w'}]_{z=z_C} + \\ & - (P_0 v_0 + Q_0 w_0) - (P_C v_C + Q_C w_C), \end{aligned} \tag{34}$$

which is a modification of functional (5). It is simple to verify that the stationary conditions of functional (34) are eqns (6) (7) (8) and the compatibility equations

$$v = v_0$$
$$w = w_0$$

in $z = 0$, and the analogous in $z = z_C$.

In this way a coherent variational formulation for the whole system, beam plus (underlying and adjacent) support can be given. On this purpose, it is sufficient to implement the relevant transitional conditions at the point of connection between consecutive parts of the support. As an example, in the case of point A, Fig. 1, the terms in the last round brackets of functional (34) are dropped, subscript C is changed with subscript A, and superscript $^-$ is imposed to $\psi_{v/v'}$ in the second square brackets. Simultaneously, superscript $^+$ is imposed to $\psi_{v/v'}$ in the first square brackets of boundary terms in functional (25).

6. FINITE ELEMENT APPLICATIONS

Functional (25) allows the development of a two-field finite element approach where the response of the support is represented through a prescribed displacement field and the stress field fulfils, *a priori*, equilibrium in the beam—see eqns (20)–(23). The related variational formulation is of a hybrid type (Pian, 1973), because interelemental equilibrium is enforced as a natural condition and is met at the solution level. On this regard, it should be noted—eqns (30), (31)—that the interelemental equilibrium conditions match the sum of axial stress and lumped force $\psi_{w/w'}$, or shear force and lumped force $\psi_{v/v'}$ for each element, so that axial and shear forces are discontinuous, in principle, for a lack of continuity in the derivative of displacement.

The approach results in an equilibrium stress finite element model and leads to a matrix-displacement procedure of analysis (Fraeijs De Veubeke, 1965). Implementing a beam finite element on this ground is a matter of routine, and it seems sufficient here to sketch the main steps of the procedure. The displacement components are represented through complete algebraic polynomials of a prescribed degree and the relevant expressions (4_2), (4_3) are obtained. The relevant axial and shear forces and bending moment for the beam, in equilibrium with the response of the support, are obtained from eqns (23). Likewise, the inner forces in equilibrium with given load patterns are obtained from eqns (22). The expressions so prepared are inserted into functional (25), which becomes a function whose variables are the stress parameters V_0, N_0, M_0, the parameters of the displacement representation, and the nodal generalized displacements of the beam element, v_A, w_A, φ_A, v_B, w_B, φ_B. Extremity loads P_A, Q_A, C_A, P_B, Q_B, C_B are regarded as generalized nodal forces of the element. Imposing the stationary condition with respect to stress and displacement parameters, leads to the discretized compatibility equations of the model. Likewise, the stationary condition of *MCE* with respect to the generalized displacements leads to the discretized equilibrium equations. Finally, elimination of stress and displacement parameters between compatibility and equilibrium equations produces the nodal generalized force–displacement relationship for the element.

The least degree for the displacement representation is one—see eqns (4), it would be zero if k_{v1}, k_{w1} were zero (Winkler's model). It should be observed that the displacement parameters can be kept independent of the nodal displacements so as to fully exploit the variational approach, since compatibility is not a prerequisite for the model. If n is the degree of the displacements w and v, taking into account that part of the stress field which depends on V_0, N_0, M_0, then the resulting indeterminate stress field is complete up to the degree $n+1$ for axial and shear forces, and $n+2$ for bending moment. Thus, the degree of the stress field in the proposed model is, as a rule, larger than the one in a compatible model with the same number of displacement parameters, thus resulting in a more accurate stress representation.

Finally, note that functional (34) allows the development of a finite element representation of the support, which is coherent with the one applied for the beam. The results obtained for some test problems are exposed, making reference to Fig. 1 and to Section 2 for the symbols.

Axially loaded beam

The beam has length $L = z_B - z_A$ and circular cross-section of diameter d. The one-parameter (Winkler's) model has been assumed for the support. All the loads on the beam are zero, except Q_B, and the extremity A is free. The solution of the problem—Scott, 1981, chapter 4—in terms of displacement, w, and axial force, N, is

$$(E_s L/Q_B)w = \beta(e^{\lambda L\zeta} + e^{-\lambda L\zeta})/e^{\lambda L} - e^{-\lambda L})$$

$$N/Q_B = (e^{\lambda L\zeta} - e^{-\lambda L\zeta})/e^{\lambda L} - e^{-\lambda L}),$$

where $\zeta = (z - z_A)/L$, $\beta = 4\rho^2/(\lambda L\pi K)$, $\lambda L = \rho[K(1+v_s)]^{-1/2}$, $\rho = L/d$, $K = E/E_s$, A and E are the cross-sectional area and Young's modulus of the beam, E_s and v_s are Young's and Poisson's moduli of the support, respectively. Winkler's modulus of the support is $k = \pi E/(4K)$.

Three equilibrium finite elements have been implemented on the variational ground of functional (25), with constant (PESEC), linear (PESEL), quadratic (PESEQ) representations of the support's displacement, respectively—the constant pattern can be employed for the displacements with Winkler's model of response. Moreover, two compatible models have been implemented, with linear (PESCL) and quadratic (PESCQ) displacements for the beam.

The values $\rho = 25$, $K = 50$, $v_s = 0$ have been employed for a test problem. The results obtained for the adimensional displacement, $w_{ad} = (E_s L/Q_B)w$, at the extremities A and B, $w_{A,ad}$ and $w_{B,ad}$ and for the adimensional axial force, $N_{ad} = N/Q_B$, at the midspan, $N_{ad}(L/2)$, of the beam are presented in the graphs of Fig. 2 and in Tables 1, 2, 3 vs the number of (equal) elements employed in the mesh. The comparison appears to be favourable for the proposed elements, both in terms of displacements and axial force.

Bent beam

The two-parameter (Vlasov–Leont'ev's) model has been assumed for the support. The stiffness coefficients:

—k_{vv}, shear force due to unit transverse displacement and null rotation
—$k_{v\varphi}$, bending moment due to unit transverse displacement and null rotation
—$k_{\varphi\varphi}$, bending moment due to unit rotation and null displacement

for the extremity A of the beam (extremity B is fixed), Fig. 1, have been evaluated.

Three equilibrium finite elements have been implemented on the variational basis of functional (25), with linear (BESEL), quadratic (BESEQ), cubic (BESEC) displacements. Moreover, the compatible, cubic displacement finite element (BESCC), presented by Zhahoua and Cook (1983), has been employed for comparison. Two cases have been considered (Chiwanga and Valsangkar, 1988; Razaqpur and Shah, 1991). In both cases shear deformation of the beam is disregarded, in the theory, by the authors of the referenced papers. On this regard, a sufficiently high numerical value has been given here to shear stiffness in order to exclude the effects of shear deformability in the results.

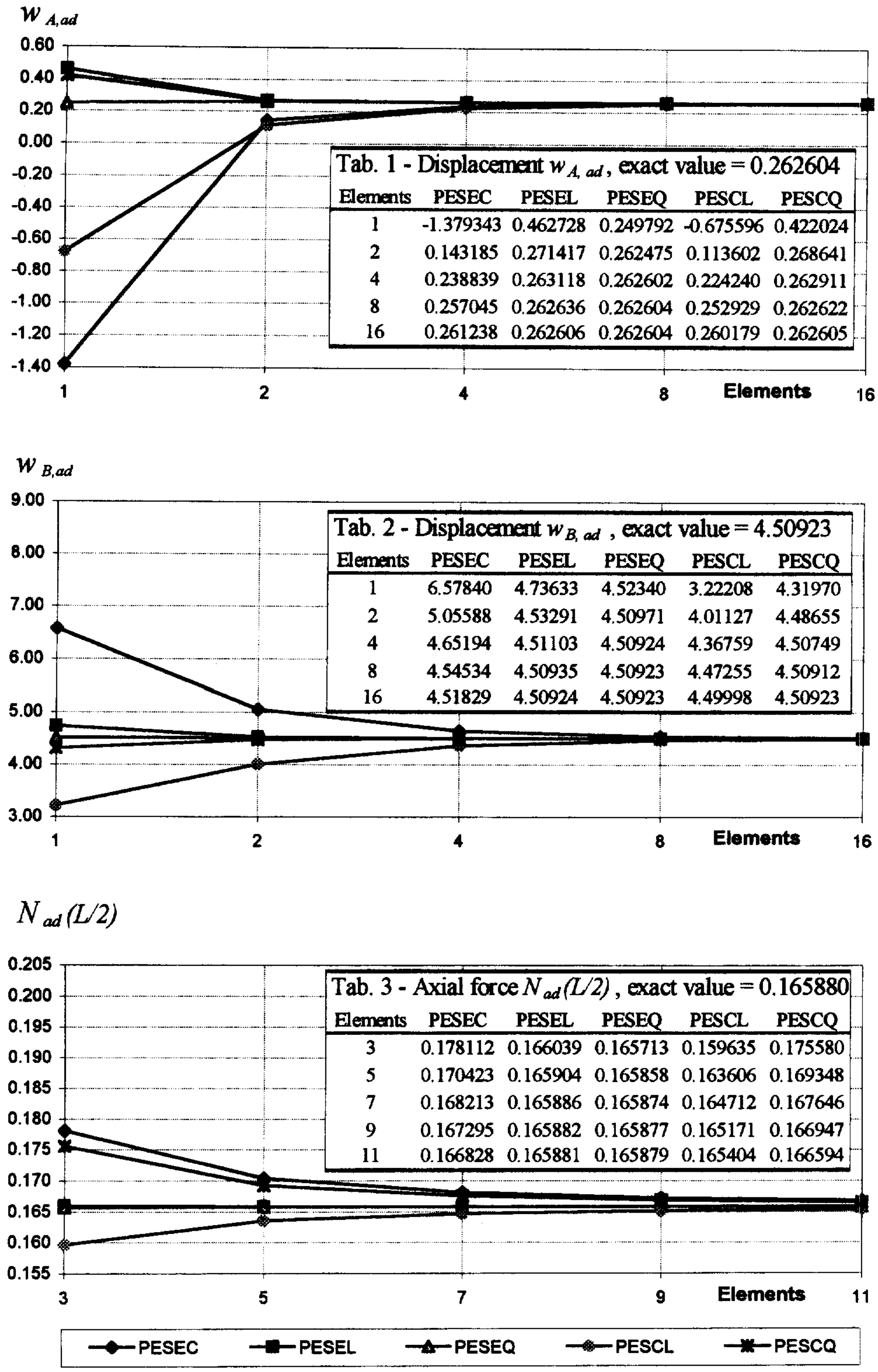
Case 1. This problem is exposed by Chiwanga and Valsangkar (1988), Example 1. The following data have been employed: $L = 5$ m, $E = 3.45 \times 10^7$ kN/m^2, $I = 4.168 \times 10^{-4}$ m^4, $k_v = 3236.5$ kN/m^2, $k_{v1} = 1298.54$ kN. The results of the analysis are presented in Fig. 3 and Tables 4, 5, 6, with respect to the number of elements employed in the discretization. The exact values of comparison have been obtained from the formulae derived from the solution of the differential equation reported by Chiwanga and Valsangkar, 1988.

Case 2. This problem is exposed by Razaqpur and Shah (1991), Example 2. The following data have been employed: $L = 5$ m, $EI = 2000$ kN $\times$ m^2, $k_v = 64$ kN/m^2, $k_{v1} = 800$ kN. The results of the analysis are presented in Fig. 4 and Tables 7, 8, 9, with respect to the number of elements employed in the discretization. The analytical solution reported by Razaqpur and Shah, 1991, has been applied to derive the exact values for comparison.

For the first case, the data result in $k_{v1} < (4k_v EI)^{1/2}$, for the second case in $k_{v1} > (4k_v EI)^{1/2}$, and different analytical developments are required for the exact solution. From the numerical results exposed, convergence seems better in the first case. In both cases, the proposed elements exhibit a satisfactory performance.

7. CONCLUSIONS

A mixed and a modified complementary energy variational formulations for the problem of a beam on a continuous elastic support have been presented and discussed. Such formulations constitute a formal correlative to the variational formulation of the problem based on the minimum potential energy principle. The second formulation provides the



Tab. 1 - Displacement $w_{A,ad}$, exact value = 0.262604

Elements	PESEC	PESEL	PESEQ	PESCL	PESCQ
1	-1.379343	0.462728	0.249792	-0.675596	0.422024
2	0.143185	0.271417	0.262475	0.113602	0.268641
4	0.238839	0.263118	0.262602	0.224240	0.262911
8	0.257045	0.262636	0.262604	0.252929	0.262622
16	0.261238	0.262606	0.262604	0.260179	0.262605

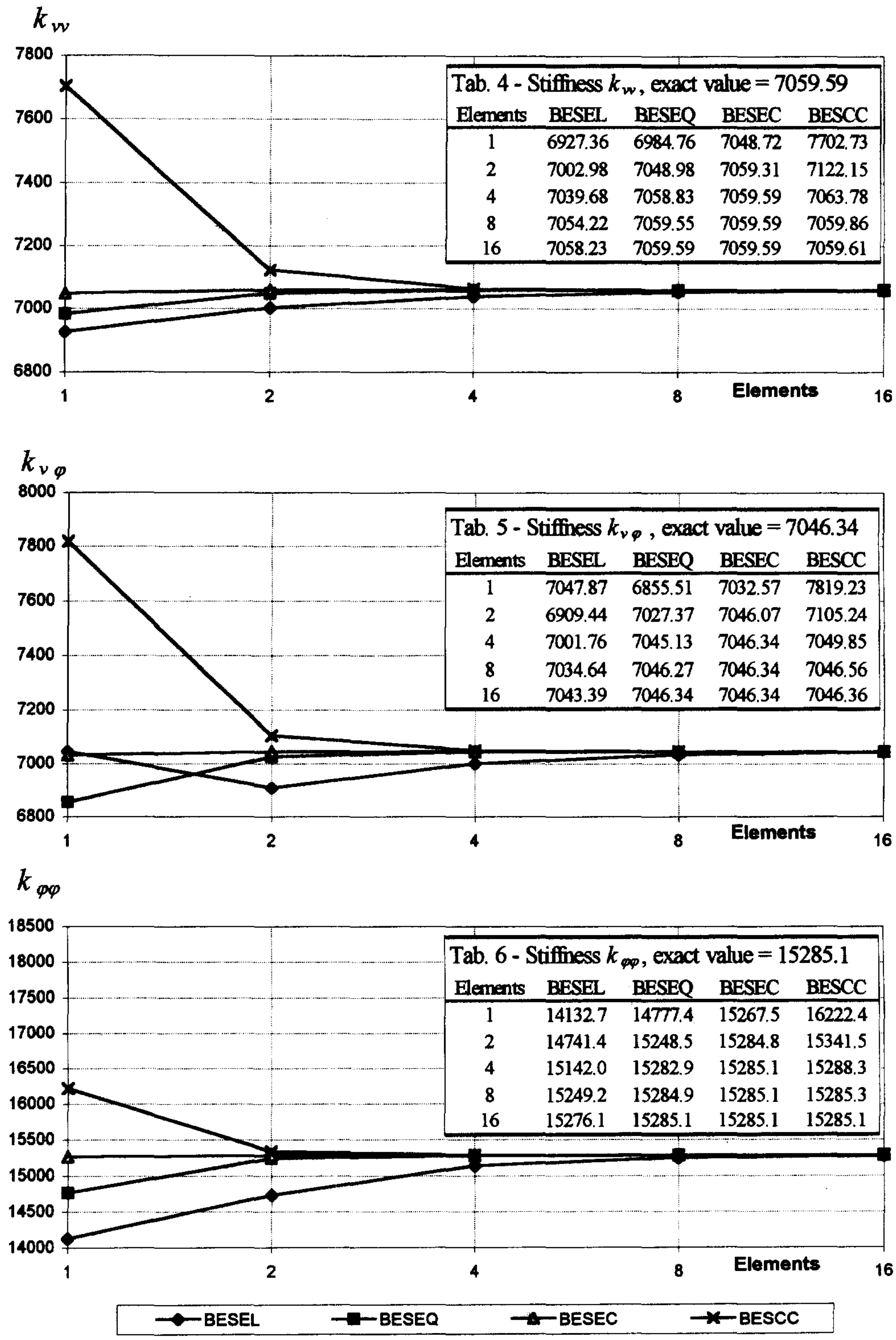
Tab. 2 - Displacement $w_{B,ad}$, exact value = 4.50923

Elements	PESEC	PESEL	PESEQ	PESCL	PESCQ
1	6.57840	4.73633	4.52340	3.22208	4.31970
2	5.05588	4.53291	4.50971	4.01127	4.48655
4	4.65194	4.51103	4.50924	4.36759	4.50749
8	4.54534	4.50935	4.50923	4.47255	4.50912
16	4.51829	4.50924	4.50923	4.49998	4.50923

Tab. 3 - Axial force $N_{ad}(L/2)$, exact value = 0.165880

Elements	PESEC	PESEL	PESEQ	PESCL	PESCQ
3	0.178112	0.166039	0.165713	0.159635	0.175580
5	0.170423	0.165904	0.165858	0.163606	0.169348
7	0.168213	0.165886	0.165874	0.164712	0.167646
9	0.167295	0.165882	0.165877	0.165171	0.166947
11	0.166828	0.165881	0.165879	0.165404	0.166594

Fig. 2. Axially loaded beam. Displacements $w_{A,ad}$, $w_{B,ad}$ at the ends, and axial force $N_{ad}(L/2)$ at midspan.



Tab. 4 - Stiffness k_{vv}, exact value = 7059.59

Elements	BESEL	BESEQ	BESEC	BESCC
1	6927.36	6984.76	7048.72	7702.73
2	7002.98	7048.98	7059.31	7122.15
4	7039.68	7058.83	7059.59	7063.78
8	7054.22	7059.55	7059.59	7059.86
16	7058.23	7059.59	7059.59	7059.61

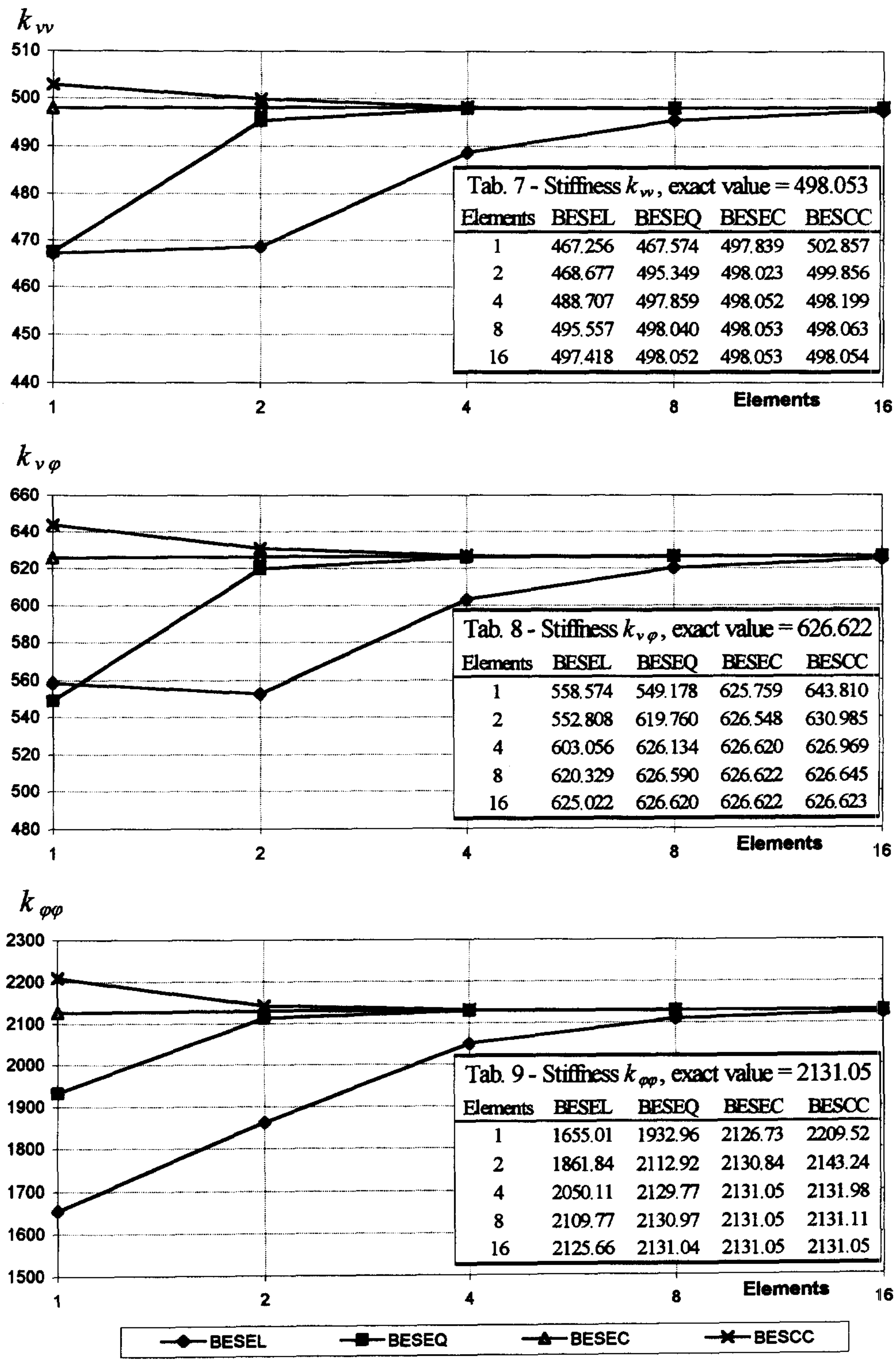
Tab. 5 - Stiffness $k_{v\varphi}$, exact value = 7046.34

Elements	BESEL	BESEQ	BESEC	BESCC
1	7047.87	6855.51	7032.57	7819.23
2	6909.44	7027.37	7046.07	7105.24
4	7001.76	7045.13	7046.34	7049.85
8	7034.64	7046.27	7046.34	7046.56
16	7043.39	7046.34	7046.34	7046.36

Tab. 6 - Stiffness $k_{\varphi\varphi}$, exact value = 15285.1

Elements	BESEL	BESEQ	BESEC	BESCC
1	14132.7	14777.4	15267.5	16222.4
2	14741.4	15248.5	15284.8	15341.5
4	15142.0	15282.9	15285.1	15288.3
8	15249.2	15284.9	15285.1	15285.3
16	15276.1	15285.1	15285.1	15285.1

Fig. 3. Bent beam. Stiffness k_{vv}, $k_{v\varphi}$, $k_{\varphi\varphi}$ at extremity A.



Tab. 7 - Stiffness k_{vv}, exact value = 498.053

Elements	BESEL	BESEQ	BESEC	BESCC
1	467.256	467.574	497.839	502.857
2	468.677	495.349	498.023	499.856
4	488.707	497.859	498.052	498.199
8	495.557	498.040	498.053	498.063
16	497.418	498.052	498.053	498.054

Tab. 8 - Stiffness $k_{v\varphi}$, exact value = 626.622

Elements	BESEL	BESEQ	BESEC	BESCC
1	558.574	549.178	625.759	643.810
2	552.808	619.760	626.548	630.985
4	603.056	626.134	626.620	626.969
8	620.329	626.590	626.622	626.645
16	625.022	626.620	626.622	626.623

Tab. 9 - Stiffness $k_{\varphi\varphi}$, exact value = 2131.05

Elements	BESEL	BESEQ	BESEC	BESCC
1	1655.01	1932.96	2126.73	2209.52
2	1861.84	2112.92	2130.84	2143.24
4	2050.11	2129.77	2131.05	2131.98
8	2109.77	2130.97	2131.05	2131.11
16	2125.66	2131.04	2131.05	2131.05

Fig. 4. Bent beam. Stiffness k_{vv}, $k_{v\varphi}$, $k_{\varphi\varphi}$ at extremity A.

basis for an equilibrium finite element approach, which seems to be, from some numerical tests, a profitable alternative to the traditional, compatible finite element approach.

Acknowledgements—The financial support of the Italian Ministry of University and Scientific Research (MURST) and of the National Council of Research (CNR, grant No. 93.02191.07) made this work possible. Numerical developments were performed at LAboratorio di Meccanica Computazionale (LAMC), DISTART, Università di Bologna, Bologna, Italy.

REFERENCES

Atluri, S. N., Tong, P. and Murakawa, H. (1983) Recent studies in hybrid and mixed finite element methods in mechanics. In *Hybrid and mixed finite element methods*, eds Atluri *et al.* John Wiley, New York, pp. 51–71.

Chin, T. K. (1970) Estimation of the ultimate load of piles from tests not carried to failure. *Proceedings of the II South East Asian Conference on Soil Engineering*, Singapore, pp. 81–90.

Chiwanga, M. and Valsangkar, A. J. (1988) Generalized beam element on two-parameter elastic foundation. *Journal of Structural Engineering, ASCE* **114**(6), 1414–1427.

Eisenberger, M. and Yankelevsky, D. Z. (1985) Exact stiffness matrix for beams on elastic foundation. *Computers and Structures* **21**(6), 1355–1359.

Eisenberger, M. and Clastornik, J. (1987) Beams on variable two-parameter elastic foundation. *Journal of Engineering Mechanics, ASCE* **113**(10), 1454–1466.

Filonenko-Borodich, M. M. (1940) Some approximate theories of the elastic foundation. *Uchenyie Zapiski Moskowskogo Gosudarstvennogo Universiteta. Mekhanica* **46,** 3–18 (in Russian).

Fraeijs De Veubeke, B. (1965) Displacement and equilibrium models in the finite element method. In *Stress Analysis*, ch. 9 eds Zienkiewicz and Holister. Wiley, NY.

Hetényi, M. (1946) *Beams on Elastic Foundations*. University of Michigan Press, Ann Arbor.

Karamanlidis, D. and Prakash, V. (1989) Exact transfer and stiffness matrices for a beam/column resting on a two-parameter foundation. *Computer Methods in Applied Mechanical Engineering* **72,** 77–89.

Kerr, A. D. (1964) Elastic and viscoelastic foundation models. *Journal of Applied Mechanics* **31,** 491–498.

Kerr, A. D. (1976) On the derivation of well posed boundary value problems in structural mechanics. *International Journal of Solids and Structures* **12**(1), 1–11.

Kerr, A. D. (1984) On the formal development of elastic foundation models. *Ingenieur Archiv* **54,** 455–464.

Liu, Q. F. and Meyerhof, G. G. (1987) New method for nonlinear analysis of rigid piles in clay. *Computers and Geotechnics* **3,** 185–212.

Miranda, C. and Nair, K. (1966) Finite elements on elastic foundation. *Journal of the Structural Division, ASCE* **92**(ST2), 131–142.

Mourelatos, Z. P. and Parsons, M. G. (1987) A finite element analysis of beams on elastic foundation including shear and axial effects. *Computers and Structures* **27**(3), 323–331.

Pasternak, P. L. (1954) *Fundamentals of a new method of analysis of structures on an elastic foundation by means of two foundation moduli.* Gosudarstvenuoe izdatel'stvo literaturi po stroitel'stvu i Arkhitecture, Moscow, USSR (in Russian).

Pian, T. H. H. (1973) Hybrid models. In *Numerical and Computer Methods in Structural Mechanics*, eds Fenves *et al.* Academic Press, New York, pp. 50–78.

Poulos, H. G. and Davis, E. H. (1980) *Pile Foundation Analysis and Design*. John Wiley, New York.

Randolph, M. F. and Wroth, C. P. (1978) Analysis of deformation of vertically loaded piles. *Journal of the Geotechnical Engineering Division*, ASCE **104**(GT12), 1465–1488.

Razaqpur, A. G. (1986) Stiffness and beam-columns on elastic foundation with exact shape functions. *Computers and Structures* **24**(5), 813–819.

Razaqpur, A. G. and Shah, K. R. (1991) Exact analysis of beams on two-parameter elastic foundations. *International Journal of Solids and Structures* **27**(4), 435–454.

Sayegh, A. F. and Tso, F. K. (1988) Finite element analysis of 3-D beam-columns on a nonlinear foundation. *Computers and Structures* **28**(6), 699–715.

Scott, R. F. (1981) *Foundation Analysis.* Prentice-Hall, Inc., Englewood Cliffs, N.J.

Selvadurai, A. P. S. (1979) *Elastic Analysis of Soil-Foundation Interaction.* Elsevier, Amsterdam.

Shirima, L. M. and Giger, M. W. (1992) Timoshenko beam element resting on two-parameter elastic foundation. *Journal of Engineering Mechanics, ASCE* **118**(2), 280–295.

Sirosh, S. N. and Ghali, A. (1989) A general finite element for beams or beam-columns with or without an elastic foundation. *International Journal of Numerical Methods Engineering* **28,** 1061–1076.

Soldatos, K. P. and Selvadurai, A. P. S. (1985) Flexure of beams resting on hyperbolic elastic foundation. *International Journal of Solids and Structures* **21**(4), 373–388.

Ting, B. Y. and Mockry, E. F. (1984) Beam on elastic foundation finite element. *Journal of Structural Engineering, ASCE* **110**(10), 2325–2339.

Vaziri, H. H. and Xie, J. (1990) A method of analysis of axially loaded piles in nonlinear soils. *Computers and Geotechnics* **10,** 149–159.

Vlasov, V. Z. and Leont'ev, N. N. (1966) *Beams, Plates and Shells on Elastic Foundations.* English translation NTIS Accession N.N67-14238.

Winkler, E. (1867) *Die Lehre von der Elastizität und Festigkeit*, Prag.

Yokoyama, T. (1988) Parametric instability of Timoshenko Beams resting on an elastic foundation. *Computers and Structures* **28**(2), 207–216.

Zhahoua, F. and Cook, R. D. (1983) Beam element on two-parameter elastic foundation. *Journal of Engineering Mechanics, ASCE* **109**(6), 1390–1402.

APPENDIX A

The first variation of functional TPE reads

$$
\begin{aligned}
\delta TPE = & \int_{s\backslash b} [\psi_{v/v} - \psi'_{v/v'} - p)\,\delta v + (\psi_{w/w} - \psi'_{w/w'} - q)\,\delta w]\,\mathrm{d}z + \\
& - [(\psi_{v/v'} + P_0)\,\delta v + (\psi_{w/w'} + Q_0)\,\delta w]_{z=0} + [(\psi_{v/v'} - P_C)\,\delta v + (\psi_{w/w'} - Q_C)\,\delta w]_{z=z_C} + \\
& + \int_b \{[(-GA_t(v'+\varphi))' + \psi_{v/v} - \psi'_{v/v'} - p]\,\delta v + [(-EAw')' + \psi_{w/w} - \psi'_{w/w'} - q)\,\delta w + \\
& + [GA_t(v'+\varphi) - (EI\varphi')']\,\delta\varphi\}\,\mathrm{d}z + \\
& + \{[\psi^-_{v/v'} - \psi^+_{v/v'} - GA_t(v'+\varphi) - P_A]\,\delta v + (\psi^-_{w/w'} - \psi^+_{w/w'} - EAw' - Q_A)\,\delta w + \\
& - (EI\varphi' + C_A)\,\delta\varphi\}_{z=z_A} + \\
& + \{[\psi^+_{v/v'} - \psi^-_{v/v'} + GA_t(v'+\varphi) - P_B]\,\delta v + (\psi^+_{w/w'} + \psi^-_{w/w'} + EAw' - Q_B)\,\delta w + \\
& + (EI\varphi' - C_B)\,\delta\varphi\}_{z=z_B}.
\end{aligned}
$$

The first variation of functional MHR reads

$$
\begin{aligned}
\delta MHR = & \int_b [(w' - N/EA)\,\delta N + ((v'+\varphi) - V/GA_t)\,\delta V + (\varphi' - M/EI)\,\delta M]\,\mathrm{d}z + \\
& + \int_b [(-V' + \psi_{v,v} - \psi'_{v/v'} - p)\,\delta v + (-N' + \psi_{w,w} - \psi'_{w,w'} - q)\,\delta w + (V - M')\,\delta\varphi]\,\mathrm{d}z + \\
& - [(V + \tilde{V}_A)\,\delta v + (N + \tilde{N}_A)\,\delta w + (M + \tilde{M}_A)\,\delta\varphi]_{z=z_A} + \\
& + [(V + \tilde{V}_B)\,\delta v + (N + \tilde{N}_B)\,\delta w + (M + \tilde{M}_B)\,\delta\varphi]_{z=z_B} + \\
& - [(v - v_A)\,\delta\tilde{V}_A + (w - w_A)\,\delta\tilde{N}_A + (\varphi - \varphi_A)\,\delta\tilde{M}_A]_{z=z_A} + \\
& + [(v - v_B)\,\delta\tilde{V}_B + (w - w_B)\,\delta\tilde{N}_B + (\varphi - \varphi_B)\,\delta\tilde{M}_B]_{z=z_B} + \\
& + (\tilde{V}_A - P_A)\,\delta v_A + (\tilde{N}_A - Q_A)\,\delta w_A + (\tilde{M}_A - C_A)\,\delta\varphi_A - [\psi_{v,v'}\,\delta v + \psi_{w,w'}\,\delta w]_{z=z_A} + \\
& - [(\tilde{V}_B + P_B)\,\delta v_B + (\tilde{N}_B + Q_B)\,\delta w_B + (\tilde{M}_B + C_B)\,\delta\varphi_B] + [\psi_{v,v'}\,\delta v + \psi_{w,w'}\,\delta w]_{z=z_B}.
\end{aligned}
$$

APPENDIX B

For the sake of clarity, the first variation of functional MCE is split into the contributions due to the variations with respect to its arguments:

$$
\begin{aligned}
\delta_{/N_0,V_0,M_0} MCE = & [w_B - w_A - \mathcal{N}(z_B)]\,\delta N_0 + [v_B - v_A + \varphi_B(z_B - z_A) - \mathcal{V}(z_B) + \mathcal{M}(z_B) - \mathcal{M}(z_B)(z_B - z)]\,\delta V_0 + \\
& + [\varphi_B - \varphi_A - \mathcal{M}(z_B)]\,\delta M_0, \qquad \text{(B1)}
\end{aligned}
$$

$$
\begin{aligned}
\delta_{/w} MCE = & -\int_b \mathcal{N}'\,\delta_{/w} N_r\,\mathrm{d}z - \int_b \delta_{/w}\psi_w\,\mathrm{d}z + \\
& - w_A\,\delta_{/w}(N_r + \psi_{w/w'})_{z=z_A} + w_B\,\delta_{/w}(N_r + \psi_{w/w'})_{z=z_B} = \\
= & -\int_b \mathcal{N}' \left[\int_{z_A}^{z} \psi_{w/ww}\,\delta w\,\mathrm{d}\zeta - (\psi_{w/w'w'}\,\delta w'|_z - \psi_{w/w'w'}\,\delta w'|_{z_A})\right]\mathrm{d}z + \\
& - \int_b (w\psi_{w/ww}\,\delta w + w'\psi_{w/w'w'}\,\delta w')\,\mathrm{d}z + \\
& + (w_B - w_A)\psi_{w/w'w'}\,\delta w'|_{z_A} + w_B \int_{z_A}^{z_B} \psi_{w/ww}\,\delta w\,\mathrm{d}\zeta = \\
& \dots \\
= & -\int_b \{[w - w_B - \mathcal{N} + \mathcal{N}(z_B)]k_w - [(w' - \mathcal{N}')k_{w1}]'\}\,\delta w\,\mathrm{d}z + \\
& + [(w_B - w_A - \mathcal{N}(z_B))(k_{w1}\,\delta w')]_{z=z_A} - (w' - \mathcal{N}')k_{w1}\,\delta w|_{z_A}^{z_B}, \qquad \text{(B2)}
\end{aligned}
$$

$$
\begin{aligned}
\delta_{/v} MCE = & -\int_b \mathscr{M}' \, \delta_{/v} M_r \, \mathrm{d}z - \int_b \mathscr{V}' \, \delta_{/v} V_r \, \mathrm{d}z - \int_b \delta_{/v} \psi_v \, \mathrm{d}z + \\
& - v_A \, \delta_{/v} (V_r + \psi_{v/v'})_{z=z_A} + v_B \, \delta_{/v} (V_r + \psi_{v/v'})_{z=z_B} + \varphi_B \, \delta_{/v} M_r |_{z=z_B} = \\
& - \int_b \left(- \mathscr{M}' \int_{z_A}^{z} \psi_{v/vv} \, \delta v \, \mathrm{d}\zeta + \mathscr{M} \psi_{v/v'v'} \, \delta v' \right) \mathrm{d}z - (\psi_{v/v'v'} \, \delta v' |_{z_A}) \int_b (\mathscr{M}'(z - z_A) \, \mathrm{d}z + \\
& - \left[\mathscr{M} \int_{z_A}^{z} \left(\int_{z_A}^{\zeta} \psi_{v/vv} \, \delta v \, \mathrm{d}\varsigma \right) \mathrm{d}\zeta \right] \Bigg|_{z_A}^{z_B} + \left(\mathscr{M} \int_{z_A}^{z} \psi_{v/v'v'} \, \delta v' \, \mathrm{d}\zeta \right) \Bigg|_{z_A}^{z_B} + \\
& - \int_b \mathscr{V}' \left[\int_{z_A}^{z} \psi_{v/vv} \, \delta v \, \mathrm{d}\zeta - (\psi_{v/v'v'} \, \delta v' |_z - \psi_{v/v'v'} \, \delta v' |_{z_A}) \right] \mathrm{d}z + \\
& - \int_b (v/\psi_{v/vv} \, \delta v + v' \psi_{v/v'v'} \, \delta v') \, \mathrm{d}z + \\
& + (v_B - v_A) \psi_{v/v'v'} \, \delta v' |_{z_A} + v_B \int_{z_A}^{z_B} \psi_{v/vv} \, \delta v \, \mathrm{d}\zeta + \\
& + \left[\int_b \left(\int_{z_A}^{z} \psi_{v/vv} \, \delta v \, \mathrm{d}\varsigma \right) \mathrm{d}\zeta - \int_b \psi_{v/v'v'} \, \delta v' \, \mathrm{d}z + \psi_{v/v'v'} \, \delta v' |_{z_A} (z_B - z_A) \right] \varphi_B = \\
& \ldots \\
= & - \int_b \{ [\mathscr{M} - \mathscr{M}(z_B) - \mathscr{V} + \mathscr{V}(z_B) - v + v_B - (\varphi_B - \mathscr{M}(z_B))(z_B - z)] k_v + \\
& + [(\mathscr{V}' - \mathscr{M} + v' - \varphi_B + \mathscr{M}(z_B)) k_{v1}]' \} \, \delta v \, \mathrm{d}z + \\
& + [(\mathscr{V}' - v' - \varphi_B) k_{v1} \, \delta v]_{z=z_B} - [(\mathscr{V}' - v' - \varphi_B + \mathscr{M}(z_B)) k_{v1} \, \delta v]_{z=z_A} + \\
& + [(\mathscr{M}(z_B) - \mathscr{V}(z_B) + v_B - v_A + (\varphi_B - \mathscr{M}(z_B))(z_B - z_A)](k_{v1} \, \delta v')_{z=z_A},
\end{aligned}
\tag{B3}
$$

$$
\delta_{/w_A, v_A \varphi_A} MCE = [(N + \psi_{w,w'} + Q_A) \, \delta w_A + (V + \psi_{v,v'} + P_A) \, \delta v_A + (M + C_A) \, \delta \varphi_A]_{z=z_A}, \tag{B4}
$$

$$
\delta_{/w_B, v_B \varphi_B} MCE = [(N + \psi_{w,w'} - Q_B) \, \delta w_B + (V + \psi_{v,v'} - P_B) \, \delta v_B + (M - C_B) \, \delta \varphi_B]_{z=z_B}. \tag{B5}
$$